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# Continuity of Metric Projections in Uniformly Convex and Uniformly Smooth Banach Spaces

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The continuity of the metric projection onto an approximatively compact set in a uniformly convex and uniformly smooth Banach space is investigated. An explicit modulus of continuity for the metric projection which depends on the directional radius of curvature at a certain point of the set is obtained. The results generalize and improve those obtained by B. O. Björnestål.

## 1. INTRODUCTION

Let $M$ be a nonempty set in a Banach space $B$. The metric projection $P$ of $B$ into $M$ is defined by $P(x) = \{y \mid y \in M,\ \inf_{m \in M} \|x - m\| = \|x - y\|\}$, $x \in B$. In our discussion we will assume that $M$ proximinal which means that $P(x) \neq \varnothing$ for every $x$ in $B$.

It is well known that when $M$ is a closed convex set in a uniformly convex Banach space (e.g., a Hilbert space), the metric projection $P$ is a singleton and continuous. See [5] for a proof.

If $M$ is a closed convex set in a Hilbert space, then (as is well known) $P$ satisfies $\|P(y) - P(x)\| \leqslant \|y - x\|$ for all $y$ and $x$. For such a set in a uniformly convex and uniformly smooth Banach space Björnestål [3] proved that $\|P(y) - P(x)\| \leqslant 2\delta^{-1}(2\tilde{\rho}(6\|y - x\|))$, where $\delta^{-1}$ is the inverse function of the modulus of uniform convexity, $\tilde{\rho}$ is the modulus of uniform smoothness and where it is assumed that $\|x - P(x)\| = 1$.

$P$ may be a singleton and continuous even if the set $M$ is not convex; as a matter of fact, Wolfe in [7] has proved that if $M$ is a $C^2$-approximately compact manifold in a Hilbert space, then $P$ is a singleton and Frechet differentiable in an open dense set. Furthermore, in this case the present author has shown in [1] that whenever, throughout an open set $S$, $P(x)$ is a singleton, continuous and $\neq x$, it is Frechet differentiable in $S$ and its Frechet derivative $P'(x)$ satisfies there $\|P'(x)\| = \rho/(\rho - r)$, where $r = \|x - P(x)\|$ and



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$\rho$ is the reciprocal of the maximum principal curvature of $M$ at $P(x)$ in the direction of $x - P(x)$. (If this principal curvature $= 0$ we take $\rho = \infty$ in which case $\|P'(x)\| = 1$.)

Even when $M$ is not $C^2$ it is possible to define the notion of directional radius of curvature, see [2, 4] for the definition, and using it we have proved the following theorems for a Hilbert space, see [2].

THEOREM 1. *Let $M$ be an approximatively compact set and suppose $P(x)$ is a singleton. Let $\rho$ be the directional radius of curvature of $M$ at $P(x)$ in the direction $x - P(x)$. We assume $x \neq P(x)$ and $r = \|x - P(x)\| \neq \rho$. Then if $m_y \in P(y)$, we have $\overline{\lim}_{y \to x}(\|m_y - P(x)\|/\|y - x\|) \leqslant 2\rho/(\rho - r)$, where $r = \|x - P(x)\|$.*

THEOREM 2. *Let $M$ be a closed convex set and assume $x \neq P(x)$. Then $\overline{\lim}_{y \to x}(\|P(y) - P(x)\|/\|y - x\|) \leqslant 2\rho/(2\rho - r)$, where $\rho$ and $r$ are defined as in Theorem 1.*

This improves the classical estimate $\|P(y) - P(x)\| \leqslant \|y - x\|$, see [4], because, as it turns out, $\rho \leqslant 0$ when $M$ is convex.

The purpose of this paper is to generalize Theorems 1 and 2 in the context of uniformly smooth and uniformly convex Banach spaces.

## 2. DEFINITIONS

A set $M$ in a Banach space $B$ is called approximatively compact if for each $x$ in $B$ and each sequence $\{m_n\} \subset M$ such that $\|x - m_n\| \to \inf_{m \in M} \|x - m\|$, there exists a subsequence converging to a point in $M$.

*Radius of Curvature.* Let $M$ be a set in a Banach space $B$. Let $x \in B$ ($x \notin M$) be a point such that $P(x)$ is a singleton. Consider the unit vector $v = (x - P(x))/(\|x - P(x)\|)$ and points $\mu$ in $M$ close to $m = P(x)$. We assume $m$ is a limit point of $M$. Next we consider the following equation in $t$:

$$|t| = \|(m + tv) - m\| = \|(m + tv) - \mu\| \qquad \text{with} \quad \mu \neq m. \tag{2.1}$$

If Eq. (2.1) does not hold for any finite $t$ we take $t = \infty$, otherwise the solution $t$ is unique.

We now define the directional radius of curvature of $M$ at $m$ in the direction $v$, $\rho(m, v)$ as

$$\rho(m, v) = \left( \overline{\lim_{\substack{\mu \to m \\ \mu \in M}}} \left\{ \frac{1}{t} : |t| = \|(m + tv) - \mu\| \right\} \right)^{-1}.$$

If (2.1) does not hold for a finite $t$ for all $\mu$ sufficiently close to $m$, we set $\rho(m, v) = \infty$. For more properties and examples of the radius of curvature see [2, 5].

## 3. The Modulus of Continuity of the Metric Projection in Uniformly Convex and Uniformly Smooth Banach Spaces

A Banach space $B$ is said to be uniformly convex if its modulus of uniform convexity $\delta$ defined by

$$\delta(\varepsilon) = \inf\{1 - \tfrac{1}{2}\|x + y\| \mid \|x\| \leqslant 1, \|y\| \leqslant 1, \|x - y\| \geqslant \varepsilon\}$$

is a strictly increasing function on [0, 2).

A Banach space is uniformly smooth if its modulus of uniform smoothness $\tilde{\rho}$ defined by

$$\tilde{\rho}(\tau) = \sup_{\substack{\|x\| = 1 \\ \|y\| = \tau}} [(\|x + y\| + \|x - y\| - 2)/2]$$

satisfies $\tilde{\rho}(\tau) = o(\tau)$ as $\tau \to 0$.

For a closed convex set $M$ in a Banach space with the above properties, B. O. Björnestål proved $\|P(y) - P(x)\| \leqslant 2\delta^{-1}(2\tilde{\rho}(6\|y - x\|))$ provided that $\|x - P(x)\| = 1$ and $y$ is sufficiently close to $x$. (Here and below $\delta^{-1}$ is the inverse function to $\delta$.) Our goal is to introduce a new technique which will allow us to estimate the modulus of continuity of the metric projection in the case that $M$ is an approximatively compact set. This technique uses the directional radius of curvature at points of the set $M$ and gives better estimates than those of Björnestål.

We shall make use of Lemma 3.1 which can be found in [6, p. 388].

LEMMA 3.1. *Let $M$ be an approximatively compact set in a Banach space $B$. Suppose $x$ in $B$ has $m$ as a unique best approximation from $M$ and let $\{x_k\}$ be any sequence converging to $x$ and $\{m_k\}$ any corresponding sequence of closest points in $M$. Then $m_k \to m$.*

We will also use Lemma 3.2 which gives an estimate on the Lipschitz continuity of the metric projection onto a sphere of radius $R$.

LEMMA 3.2. *Let $P$ be the metric projection onto the sphere centered at* 0 *and of radius $R$ in a uniformly convex Banach space. Then if $x \neq 0$ and $y \neq 0$, we have*

$$\|P(y) - P(x)\| \leqslant \frac{2R}{\|x\|} \|y - x\|.$$

*Proof.* It is obvious that for each $x \neq 0$, $P(x) = (R/\|x\|)\,x$. Then

$$\begin{aligned}
\|P(y) - P(x)\| &= \left\| \frac{R}{\|y\|}\, y - \frac{R}{\|x\|}\, x \right\| = \frac{R}{\|x\|\,\|y\|} \left\| \|x\|\, y - \|y\|\, x \right\| \\
&\leqslant \frac{R}{\|x\|\,\|y\|} \left\| \|x\|\, y - \|y\|\, y + \|y\|\, y - \|y\|\, x \right\| \\
&\leqslant \frac{R}{\|x\| \cdot \|y\|} \left\| (\|x\| - \|y\|)\, y \right\| + \frac{R}{\|x\|\,\|y\|} \|y\| \cdot \|y - x\| \\
&\leqslant \frac{R\,\|y - x\|}{\|x\|} + \frac{R\,\|y - x\|}{\|x\|} = \frac{2R\,\|y - x\|}{\|x\|}.
\end{aligned}$$

Theorem 3.1 gives a sharp estimate of the modulus of continuity of the metric projection for the case that $M$ is an approximatively compact set in a uniformly convex and uniformly smooth Banach space. Theorem 3.1 improves and generalizes results by Björnestål in [3] who obtained estimates of the modulus of continuity for the case that the set $M$ is closed and convex. Our method of proof is different from his and is based on the concept of directional radius of curvature and thus may be called the "curvature method."

THEOREM 3.1. *Let $M$ be an approximatively compact set in a uniformly convex and uniformly smooth Banach space $B$. Let $x \notin M$ be a point in $B$ which has a unique best approximation $m$ in $M$. Assume that $\rho(m, v) \neq r = \|x - m\|$. Choose $a > 0$ so that $\alpha = (\rho(m, v) - a)/\|x - m\| \notin [0, 1]$ ($a < \rho(m, v)$ if $\rho(m, v) > 0$). Let $y$ in $B$ satisfy $\|y - x\| \leqslant \|x - m\|$. Set $l = \|\alpha x - m\|/\|y - \alpha x\|$ ($y - \alpha x$ cannot vanish). Then if $y$ is sufficiently close to $x$, we have*

$$\|m_y - m\| \leqslant \|y - m\|\, \delta^{-1} \left( \frac{2\,\|z - y\|}{\|y - m\|}\, \tilde{\rho} \left( \frac{2\alpha}{\alpha - 1} \frac{\|y - x\|}{\|z - y\|} \right) \right) + \frac{2\alpha}{\alpha - 1} \|y - x\|,$$

*for $l \leqslant 2$,*

$$\|m_y - m\| \leqslant \frac{l}{2} \|y - m\|\, \delta^{-1} \left( \frac{2\,\|z - y\|}{\|y - m\|}\, \tilde{\rho} \left( \frac{2\alpha}{\alpha - 1} \frac{\|y - x\|}{\|z - y\|} \right) \right) + \frac{2\alpha}{\alpha - 1} \|y - x\|.$$

*for $l > 2$, where $m_y$ is any point of $P(y)$, and $z$ is the intersection of the line through $m + (\rho(m, v) - a)v$ and $y$ with the sphere of radius $|\rho(m, v) - a|$*

*centered at* $m + (\rho(m, v) - a)v$. *Here* $\delta$ *and* $\rho$ *are, respectively, the moduli of uniform convexity and uniform smoothness of* $B$.

*Proof.* By performing an appropriate translation we may assume that $m = 0$. Also we will treat the case when $\rho(m, v) > 0$. (The proof when $\rho(m, v) \leqslant 0$ is similar.) Now if $y$ is sufficiently close to $x$, we use Lemma 3.1 and [2, 3.1] to obtain that

$$\rho(m, v) - a = \|(\rho(m, v) - a)v\| \leqslant \|(\rho(m, v) - a)v - m_y\|,$$

where $m_y$ is any element of $P(y)$. Also observe that the unequality $\|y - m_y\| \leqslant \|y\|$ holds trivially because we assumed that 0 was in $M$.

Let $w$ be the intersection of the ray from $(\rho(m, v) - a)v$ through $y$ with the sphere of radius $\|y\|$ centered at $y$. We consider the two cases $l \leqslant 2$ and $l > 2$. See Fig. 1.

First let us assume that $l \leqslant 2$. Set $z = \alpha x + l(y - \alpha x)$; then $\|z - 2y + m_y\| = \|(l - 1)(y - \alpha x) + m_y - y\| = \|(l - 1)(\alpha x - y) + y - m_y\| \geqslant (l - 1)\|\alpha x - y\| + \|y - z\|$ because $m_y$ is outside the closed ball of radius $(l - 1)\|\alpha x - y\| + \|y - z\|$ centered at $(l - 1)(\alpha x - y) + y$. It then easily follows that $\|z - 2y + m_y\| \geqslant (l - 1)\|\alpha x - y\| + \|y - z\| = ((\|\alpha x\|/\|y - \alpha x\|) - 1)\|\alpha x - y\| + \|y - z\| = \|\alpha x\| - \|\alpha x - y\| + \|y - z\| = 2\|y - z\|$.

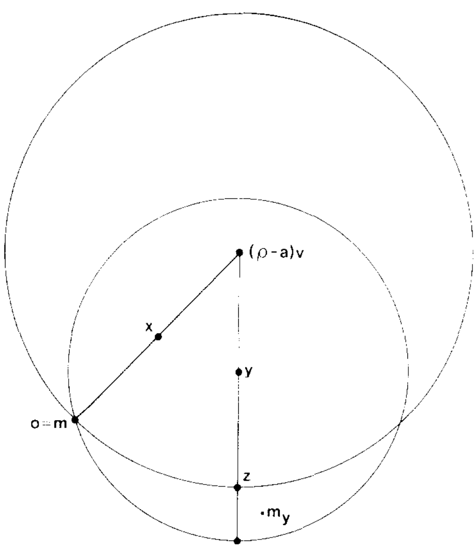


FIGURE 1

Now by the definition of modulus of uniform convexity we have

$$\frac{1}{2}\|z-y+m_y-y\| \leqslant \|y-m_y\|\left(1-\delta\left(\frac{\|m_y-z\|}{\|y-m_y\|}\right)\right)$$

from which we obtain

$$\begin{aligned}\|y-z\| \leqslant \tfrac{1}{2}\|z-y+m_y-y\| \leqslant \|y-m_y\| \\ \times\left(1-\delta\left(\frac{\|m_y-z\|}{\|y-m_y\|}\right)\right) \leqslant \|y\|\left(1-\delta\left(\frac{\|m_y-z\|}{\|y\|}\right)\right).\end{aligned}$$

The last inequality can be rewritten as

$$\delta\left(\frac{\|m_y-z\|}{\|y\|}\right) \leqslant \frac{\|y\|-\|y-z\|}{\|y\|} = \frac{\|z-w\|}{\|y\|}$$

from which we get

$$\|m_y-z\| \leqslant \|y\|\,\delta^{-1}\left(\frac{\|z-w\|}{\|y\|}\right). \tag{1}$$

In the case where $l>2$ we take a convex combination of $z-y$ and $m_y-y$ and obtain the following inequality:

$$\begin{aligned}&\left\|\frac{1}{l}(z-y)+\left(1-\frac{1}{l}\right)(m_y-y)\right\| \\ &\quad=\left\|\left(\frac{1}{l}-1\right)\alpha x+\left(1-\frac{1}{l}\right)m_y\right\| \\ &\quad=\left(1-\frac{1}{l}\right)\|\alpha x-m_y\| \geqslant \left(1-\frac{1}{l}\right)\|\alpha x\| \\ &\quad=\left(1-\frac{\|y-\alpha x\|}{\|\alpha x\|}\right)\|\alpha x\| = \|\alpha x\|-\|y-\alpha x\| \\ &\quad=\|y-z\|.\end{aligned} \tag{2}$$

In the derivation of this inequality we used the fact that $P(\alpha x)=0$. Observe also that since $\|y-z\| \leqslant \|y-m_y\|$, we obtain

$$\left\|\frac{2}{l}(z-y)+\left(1-\frac{2}{l}\right)(m_y-y)\right\| \leqslant \|m_y-y\| \leqslant \|y\|.$$

Then we apply the modulus of uniform convexity to the vectors $m_y - y$ and $(2/l)(z - y) + (1 - (2/l))(m_y - y)$ to obtain

$$\left\| \frac{1}{l}(z-y) + \left(1 - \frac{1}{l}\right)(m_y - y)\right\| \leqslant \|m_y - y\| \left(1 - \delta\left(\frac{2}{l}\frac{\|m_y - z\|}{\|m_y - y\|}\right)\right).$$

The last inequality and (2) imply

$$\|y - z\| \leqslant \|y\| \left(1 - \delta\left(\frac{2}{l}\frac{\|m_y - z\|}{\|y\|}\right)\right),$$

i.e.,

$$\delta\left(\frac{2}{l}\frac{\|m_y - z\|}{\|y\|}\right) \leqslant 1 - \frac{\|y - z\|}{\|y\|} = \frac{\|y\| - \|y - z\|}{\|y\|} = \frac{\|z - w\|}{\|y\|}$$

from which we obtain

$$\|m_y - z\| \leqslant \frac{l}{2}\|y\|\,\delta^{-1}\left(\frac{\|z - w\|}{\|y\|}\right).$$

Now we use the modulus of uniform smoothness to obtain

$$\frac{\|z - y + z\| + \|z - y - z\| - 2\|z - y\|}{2\|z - y\|} \leqslant \tilde{\rho}\left(\frac{\|z\|}{\|z - y\|}\right).$$

Observe that the closed ball of radius $\|y - z\|$ centered at $y$ is contained in the closed ball of radius $\|\alpha x\|$ centered at $\alpha x$. Note also that 0 and $z$ lie in the last mentioned ball so that $2z$ must lie outside it. Hence, $\|y - z\| < \|2z - y\|$, so that

$$\frac{\|z - w\|}{2\|z - y\|} = \frac{\|y\| - \|z - y\|}{2\|z - y\|} \leqslant \tilde{\rho}\left(\frac{\|z\|}{\|z - y\|}\right). \tag{3}$$

By Lemma 3.2

$$\|z\| \leqslant \frac{2\|\alpha x\|}{\|\alpha x - x\|}\|y - x\| = \frac{2\alpha}{\alpha - 1}\|y - x\|. \tag{4}$$

We combine inequalities (1), (3), (4) to obtain

$$\|m_y - z\| \leqslant \|y\|\,\delta^{-1}\left(\frac{2\|z - y\|}{\|y\|}\tilde{\rho}\left(\frac{2\alpha}{\alpha - 1}\frac{\|y - x\|}{\|z - y\|}\right)\right) \qquad \text{when } l \leqslant 2,$$

and

$$\|m_y - z\| \leqslant \frac{l}{2} \|y\| \delta^{-1} \left( \frac{2\|z-y\|}{\|y\|} \tilde{\rho} \left( \frac{2\alpha}{\alpha - 1} \frac{\|y-x\|}{\|z-y\|} \right) \right) \qquad \text{for } l > 2.$$

Finally, since $\|m_y\| \leqslant \|m_y - z\| + \|z\|$, we have

$$\|m_y\| \leqslant \|y\| \delta^{-1} \left( \frac{2\|z-y\|}{\|y\|} \tilde{\rho} \left( \frac{2\alpha}{\alpha - 1} \frac{\|y-x\|}{\|z-y\|} \right) \right) + \frac{2\alpha}{\alpha - 1} \|y-x\| \qquad \text{for } l \leqslant 2,$$

$$\|m_y\| \leqslant \frac{l}{2} \|y\| \delta^{-1} \left( \frac{2\|z-y\|}{\|y\|} \tilde{\rho} \left( \frac{2\alpha}{\alpha - 1} \frac{\|y-x\|}{\|z-y\|} \right) \right) + \frac{2\alpha}{\alpha - 1} \|y-x\| \qquad \text{for } 2 < l.$$

COROLLARY 3.1. *If $M$ is a closed subspace in a uniformly convex and uniformly smooth Banach space, then*

$$\|P(y) - P(x)\| \leqslant \|y - P(x)\| \delta^{-1} \left( \frac{2\|y-z\|}{\|y-P(x)\|} \tilde{\rho} \left( \frac{2\|y-x\|}{\|y-z\|} \right) \right) + 2\|y-x\|,$$

*where $z$ is the metric projection of $y$ onto the linear subspace*

$$X_\perp = \left\{ v : \frac{d}{dt} \|x - P(x) + tv\| \Big|_{t=0} = 0 \right\}.$$

*Proof.* Replace the sphere of radius $\|\alpha x\|$ centered at $\alpha x$ by $X_\perp$ and proceed as in the proof of Theorem 3.1.

Let us compare Corollary 3.1 with Björnestål's result mentioned in our Introduction. Take $y$ close to $x$, $\|x\| = \|y\| = \|y - z\| = 1$, $P(x) = 0$. We obtain $\|P(y) - P(x)\| \leqslant \delta^{-1}(2\tilde{\rho}(2\|y-x\|)) + 2\|y-x\|$ which compares favorably with Björnestål's bound $\|P(y) - P(x)\| \leqslant 2\delta^{-1}(2\tilde{\rho}(6\|y-x\|))$.

*Remark.* Corollary 3.1 can be considered a limiting case of Theorem 3.1 as $\alpha \to \infty$.

Now we would like to specialize our results to the $L^p$ spaces in order to get specific estimates.

COROLLARY 3.2. *In $L^p(\mu)$ with the same hypotheses as in Theorem 3.1, and the assumption that $\rho(m, v)/\|x - m\| \notin [1, 2]$, we have*

$$\overline{\lim_{y\to x}}\frac{\|m_y - m\|}{\|y-x\|^{p/2}} \leqslant \frac{4r^{(2-p)/2}}{\sqrt{p(p-1)}}\left(\frac{2\rho(m,v)}{\rho(m,v)-r}\right)^{p/2} \qquad \text{for } 1 < p < 2,$$

$$\overline{\lim_{y\to x}}\frac{\|m_y - m\|}{\|y-x\|^{2/p}} \leqslant 2r^{(p-2)/p}\sqrt{p(p-1)}\left(\frac{2\rho(m,v)}{\rho(m,v)-r}\right)^{2/p} \qquad \text{for } 2 < p < \infty.$$

*Proof.* Using the known moduli of uniform convexity and uniform smoothness in $L^p(\mu)$, $1 < p \leqslant 2$, see [4], we estimate $\overline{\lim}_{y\to x}\|m_y - m\|/\|y-x\|^{p/2}$ using Theorem 3.1. We then let $a \to 0$ and obtain the corollary.

We would like to point out that the estimates of this corollary are sharp in some sense because there is a subspace of $L^p$, $1 < p < 2$,. and a point $x$ such that $\overline{\lim}_{y\to x}(\|p(y) - P(x)\|/\|y-x\|^{p/2}) = C > 0$. See [3] for details.

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